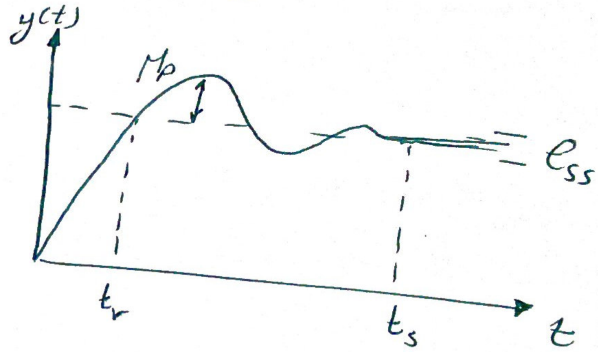
# Digital Control (Session 1)

## * Review (Control Engineering)

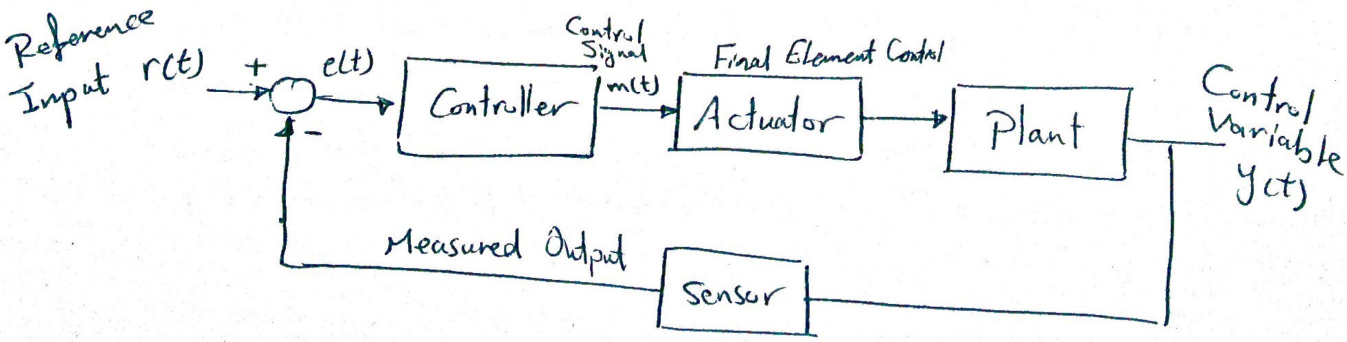
- Control theory is the general framework for studying dynamical Systems
- Control Systems objective is to improve the behaviour of the System to meet design specs (improve dynamics - reduce error)
- Any Control System Response Can be Characterized by 4-parameters:

  $t_s$ : settling time
  $t_r$ : rise time
  $M_p$ : Maximum Overshoot
  $e_{ss}$ : Steady State Error

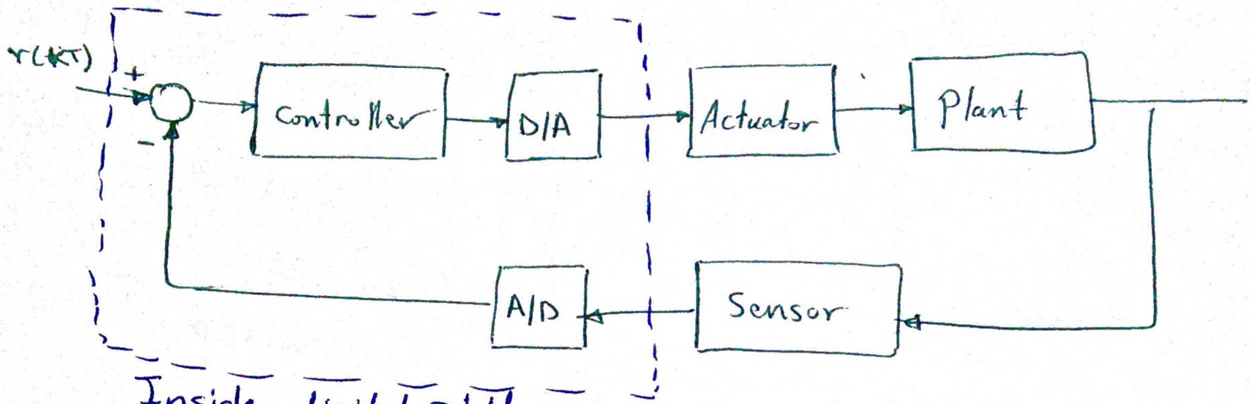



- the Phases of Control System design
  - Plant Modeling (differential Equations - Transfer function)
  - System Analysis (Root Locus - Frequency Response)
  - Controller design (PID - Lead/Lag Compensator - State feedback)
  - Implementation (Analog Components (op-amps + RLC circuits))

- the Block diagram of a Feedback Control System



* Digital Control System Structure




Inside digital platform
- Microcontroller
- DSP
- Computer
- PLC

A/D : Analog to Digital Converter

D/A : Digital to Analog Converter

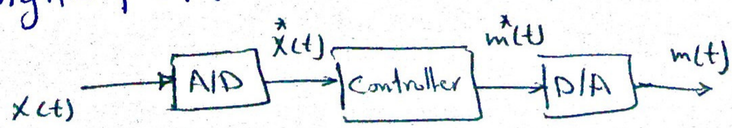
* Digital platforms are the most powerful Computing platforms specially with todays high speed of processors ( much more that speed of Systems to be Controlled).

* Digital Control Benefits:
  - Control Algorithm is Converted to a Code on a Computer System
  - More powerful to Implement Complex Control Algorithms ( there are some Control algorithm that Can be implemented only using digital control techniques)
  - Reduce the need of analog Components (affected by noise).
  - More effecient to be modified and Scaled
  - More Robust to environment disturbance

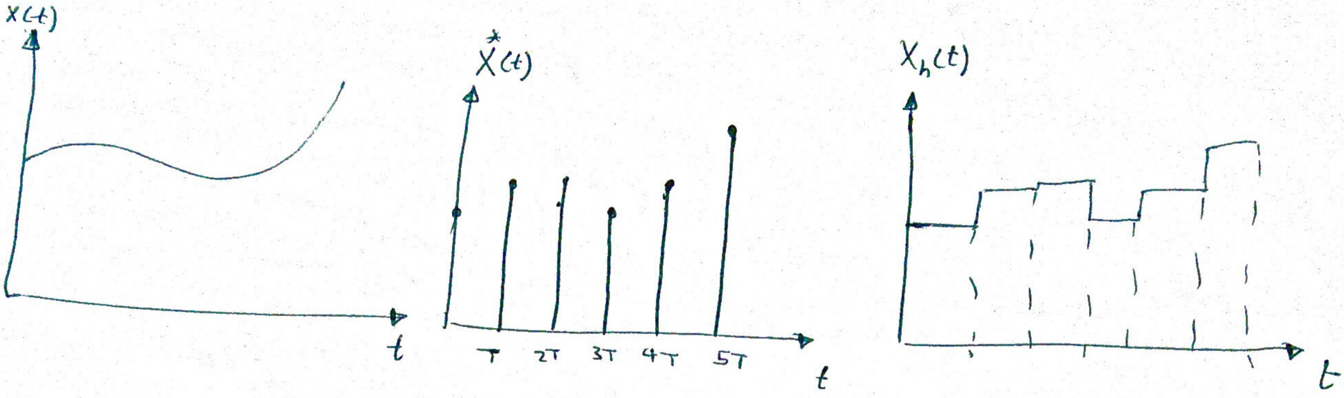
* Digital Control challenges:
  - Sampling Rate
  - Reconstruction of original Signals

As physical Systems are Continuous by nature we approximate them by discretization

* Digital part :



## * Sampling and Reconstruction




$X(t)$ : Original Analog Signal
$\overset{*}{X}(t)$ : Sampled Signal
$X_h(t)$ : Reconstructed Signal

$X_h(t)$ : the reconstructed Signal using Holder Circuit is a ladder Signal. It can be Smothed using Low Pass Filter

We note that

$$\overset{*}{X}(t) = \sum_{k=0}^{\infty} \delta(t-kT)\, x(t) \quad , \; T: \text{Sampling Period}$$

$$= \delta(t)\, x(0) + \delta(t-T)\, x(T) + \ldots$$

$$= x(0) + x(T) + x(2T) + \ldots$$

Remember Transfer Functions descriping Systems using Laplace Transform

$$\mathcal{L}[\overset{*}{X}(t)] = \sum_{k=0}^{\infty} X(kT)\, e^{-kTS} \quad (\text{Integration becomes Summation})$$

Note : this is the definition of Z-transform

$$Z = e^{kTS} \quad (Z: \text{advance operator},\; Z^{-1}: \text{delay operator})$$

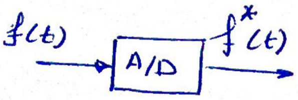
∴ we need to make revision on Z-transform

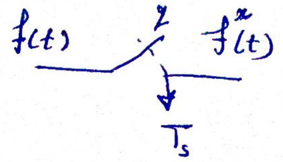
* Z-transform represents discrete Systems Transfer function
* Difference equations are solved using Z-transform

## * Sampling Process :-

→ Is there a relation between Sampling frequency and Signal frequency?

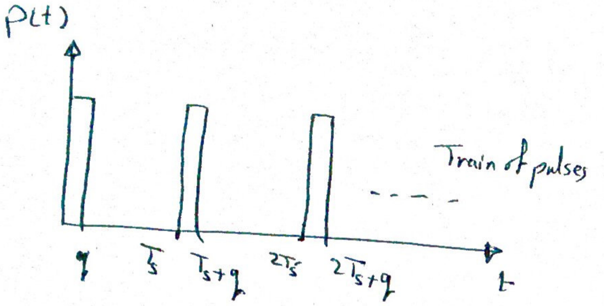
For A/D Converter :




$(q << T_s)$

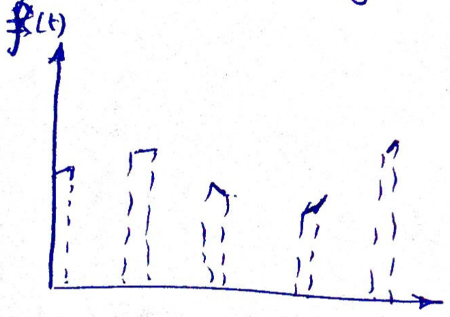
($q = 0$ for Ideal Sampler)



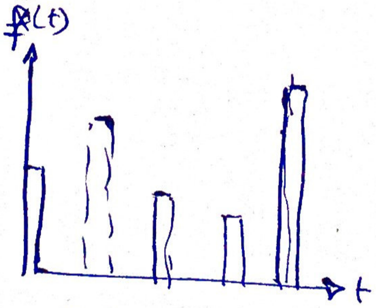

$p(t)$ : models the openning / Closing of Sampler switch

then, $x^*(t)$ is the result of multiplying $x(t)$ by $p(t)$

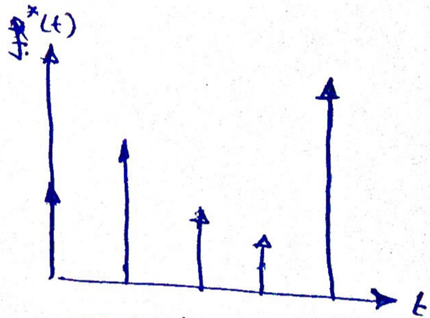
* The Sampler is Considered as an amplitude Modulation device with $p(t)$ as the Carrier Signal, $f(t)$ the input an $f^*(t)$ is the output signal.




Sampled Signal




Flat approximation




Ideal sampler

$$\boxed{f^*(t) = p(t)\, f(t)} \qquad (q << T_s)$$

where

$p(t)$ : periodic function

$$p(t) = \sum_{K=-\infty}^{\infty} U(t - KT) - U(t - (KT + q))$$

$q$ : $t_{on}$

$T_s$ : Sampling Period

* The relation between $f_s$ and $f_o$

$f_s$: Sampling frequency

$f_o$: Signal frequency

We use Fourier Analysis to derive the relation between $f_s$ and $f_o$

1. Express $p(t)$ using fourier Series ( $p(t)$ : periodic)

$$p(t) = \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_s t)$$

$$C_n = \frac{1}{T} \int_0^T p(t) \exp(-jn\omega_s t)\,dt$$

$$\therefore \quad f^*(t) = f(t) \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_s t)$$

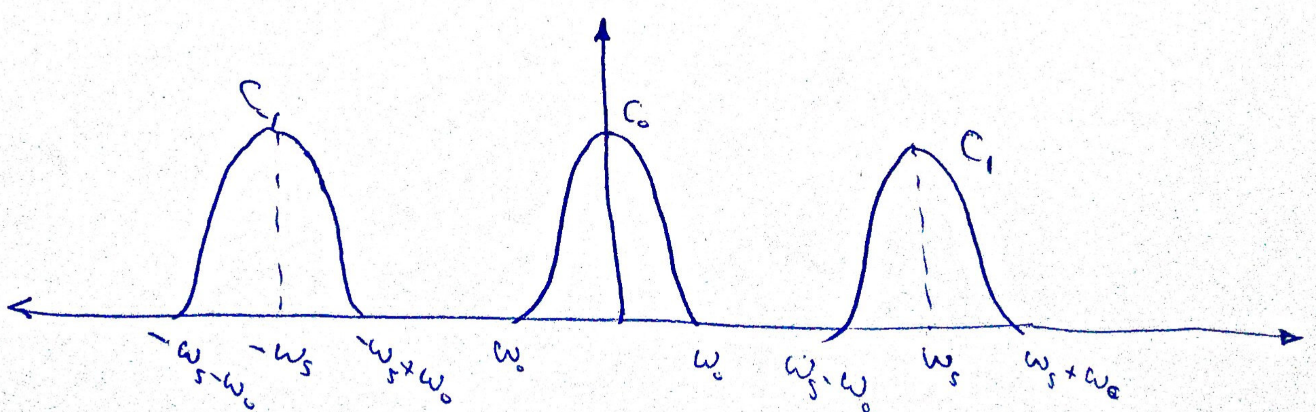
2. Taking Fourier transform of $f^*(t)$ (Sampled Signal)

$$F(f^*(t)) = F^*(j\omega) = F\left[f(t) \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_s t)\right]$$

$$= \sum_{n=-\infty}^{\infty} C_n F\left(\exp(jn\omega_s t)\, f(t)\right)$$

but $\exp(jn\omega_s t)\, f(t) \xrightarrow{F} F(j\omega - jn\omega_s)$

$$\therefore \boxed{F^*(j\omega) = \sum_{n=-\infty}^{\infty} C_n F(j\omega - jn\omega_s)}$$



From Spectrum of discrete (Sampled Signal): 

① if $\omega_s > 2\omega_o$

we LPF $\longrightarrow$ Restore $f(t)$ ✓

② if $\omega_s = 2\omega_o$

Critical case for restoring $f(t)$

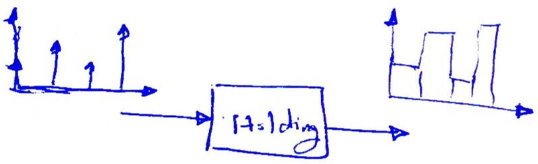
③ if $\omega_s < 2\omega_o$

we Cann't Reconstruct the Signal

> Shannon's Theory:
> For right sampling Process $\omega_s$ (sampling frequency) must be equal or larger than twice $\omega_o$
> $\omega_s \geq 2\omega_o$

∴ In Practical $\omega_s$ is choosen to be $(5-10)*\omega_o$

## * Holding Process




$$f_K(t) = f(KT) + f'(KT)(t-KT) + f''(KT)\frac{(t-KT)^2}{2!}$$

Tylor Expansion

where

- $f_K(t)$: The function expression in between $(KT)$ and $(KT+T)$
- $f(KT) = f(t)\big|_{t=KT}$ , $f'(KT) = \frac{d}{dt}f(t)\big|_{t=KT}$

→ If we choose only first term $\longrightarrow$ $f_K(t) = f(KT)$ : Zero Order Hold

Note: $\delta(t) \longrightarrow \boxed{ZOH} \longrightarrow U(t) - U(t-T)$

$$\therefore G_{ZOH}(s) = \frac{O/P(s)}{I/P(s)} = \frac{\frac{1}{s} - e^{-Ts}\cdot\frac{1}{s}}{1} = \boxed{\frac{1-e^{-Ts}}{s}}$$